J. Math. Anal. Appl. 378 (2011) 133–139

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A commutator approach to absolute continuity for unbounded Jacobi operators

Joanne Dombrowski

Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, United States

ARTICLE INFO

Article history:
Received 22 February 2010
Available online 18 January 2011
Submitted by Steven G. Krantz

Keywords:
Jacobi operators
Absolute continuity
Commutator equations

ABSTRACT

This paper uses commutator equations to study the absolute continuity of spectral measures associated with certain subclasses of unbounded self-adjoint Jacobi matrix operators determined by properties of the diagonal and subdiagonal sequences. If the diagonal sequence is the zero sequence, properties of the difference sequence of the subdiagonal determine the choice of a bounded operator for the commutator equation. The structure of the resulting commutator leads to results on absolute continuity.

1. Introduction

Consider the general class of tridiagonal matrix operators defined by two sequences $\{a_n\}$ and $\{b_n\}$ with basic assumptions $a_n > 0$, $\lim_{n\to\infty} a_n = \infty$, and b_n real. Such operators, known as Jacobi operators, have the following form on the indicated maximal domain:

$$C = \begin{bmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ 0 & 0 & a_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

$$D_C = \{x \in \ell^2\colon Cx \in \ell^2\}.$$

Various sufficient conditions are known for self-adjointness. An operator of this type will be self-adjoint if Carleman's condition $\sum \frac{1}{a_n} = \infty$ holds. Also, Berezanskii [1] showed that C is self-adjoint if either $\{a_{n-1} + b_n + a_n\}$ or $\{a_{n-1} - b_n + a_n\}$ is a bounded sequence. Additional criteria for self-adjointness are presented in [6], from which it follows, for example, that if for all n, $b_n = 0$, and $a_{2n} = a_{2n-1}$ then C is self-adjoint. This paper presents results on the spectral properties of subclasses of unbounded self-adjoint Jacobi operators determined by properties of the difference sequence associated with the subdiagonal.

If $\{\phi_n\}$ denotes the standard basis for ℓ^2 then $a_n > 0$ implies that ϕ_1 is a cyclic vector for the corresponding operator. It is worth noting that every cyclic self-adjoint operator defined on a separable Hilbert space has a tridiagonal matrix representation with respect to the basis generated by the cyclic vector. It should therefore be possible to model different types of spectral behavior within this class. In this paper results will be presented for weights of the form $a_n = n + c_n$ so that the operators are unbounded and self-adjoint. Conditions are imposed on the difference sequence $d_n = a_n - a_{n-1}$,

E-mail address: joanne.dombrowski@wright.edu.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.11.062

such as $d_{2n+2} - d_{2n} = d_{2n+1} - d_{2n-1}$, to obtain results on absolute continuity. Commutator equations play an important role in obtaining these results. There are a number of papers in the literature, employing a variety of techniques, which study the relationship between the weight sequences and the spectral properties of the corresponding operator in both the bounded and unbounded cases. See, for example, [3,5,8,9,11–13,15,16]. The results in this paper focus on the issue of absolute continuity. The operators considered in Theorem 4.5 provide a boundary case, in a sense to be clarified below, between absolute continuity and a pure discrete spectrum.

The first section of this paper establishes the overall structure needed to obtain the main results from appropriate commutator equations. The technical lemmas are of some interest in their own right. Since the operators to be studied are self-adjoint it follows from the spectral theorem that each matrix operator C is unitarily equivalent to a multiplication operator M_x: $D \to \mathcal{L}^2(\mu)$ defined on a dense subset of D of $L^2(\mu)$ by $M_x : f \to xf(x)$. If $C = \int \lambda\, dE_\lambda$ is the spectral decomposition of C then for any Borel subset β of $\mathbb{R}$, $\mu(\beta) = \|E(\beta)\phi_1\|^2$ where ϕ_1 is the first standard basis vector in ℓ^2. The standard basis vectors $\{\phi_n\}$ in ℓ^2 correspond to a sequence of polynomials in $\mathcal{L}^2(\mu)$ defined by

$$P_1(x) = 1, \qquad P_2(x) = \frac{x - b_1}{a_1},$$
$$P_{n+1}(x) = \frac{(x - b_n)P_n(x) - a_{n-1}P_{n-1}(x)}{a_n}.$$

If the self-adjoint Jacobi matrix operator C has a zero diagonal then the corresponding polynomials $\{P_n\}$ satisfy the condition $P_n(-x) = (-1)^{n+1}P_n(x)$. This symmetry will be useful in the results that follow.

2. Commutator equations

Commutator equations can be used to study the spectral measure of a self-adjoint operator. This approach has its roots in the work of C.R. Putnam [14], who was particularly interested in studying the spectral properties of the real and imaginary parts of bounded hyponormal operators. Such operators serve as examples for the following theorem.

Theorem. *Let A and B be bounded self-adjoint operators defined on a separable Hilbert space such that $AB - BA = -iK$ where $K \geqslant 0$ or $K \leqslant 0$. Let M be the smallest subspace containing the range of K and reducing A and B. Then the restriction of A (and B) to the subspace M is absolutely continuous.*

In particular, the above theorem can be used to show that the real and imaginary parts of the unilateral shift operator are absolutely continuous.

This commutator approach to the study of absolute continuity has been generalized by the author in previous work to the study of unbounded self-adjoint operators that satisfy a commutator equation on a strategic set of vectors. In the context of this paper, the key idea for establishing the absolute continuity of the spectral measure of the Jacobi operator C on an interval I is the following theorem.

Theorem. *Let C be a cyclic self-adjoint operator with cyclic vector ϕ_1, and let I be an interval. Suppose there exists a bounded self-adjoint operator J and positive constants q and Q such that if $C = \int \lambda\, dE_\lambda$, then for any bounded subinterval Δ of I,*

$$\langle JE(\Delta)\phi_1, CE(\Delta)\phi_1\rangle - \langle CE(\Delta)\phi_1, JE(\Delta)\phi_1\rangle = -i\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle$$

and

$$\bigl(Q|\Delta|\bigr)\bigl\|E(\Delta)\phi_1\bigr\|^2 \geqq \bigl|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle\bigr| \geqq q\bigl\|E(\Delta)\phi_1\bigr\|^4$$

where $|\Delta|$ denotes the Lebesgue measure of Δ. Then the spectral measure of C is absolutely continuous on I.

Proof. Since ϕ_1 is a cyclic vector for C, the spectral measure of C is given by $\mu(\beta) = \|E(\beta)\phi_1\|^2$ for any Borel set β. If Δ is a bounded subinterval of I and $\|E(\Delta)\phi_1\|^2 \neq 0$, it follows from the given inequalities that $\mu(\Delta) = \|E(\Delta)\phi_1\|^2 \leqslant \frac{Q}{q}|\Delta|$. This inequality can then be extended to Borel subsets of I, from which the result follows. □

Thus the goal is to define an appropriate operator J and to establish the required inequalities for the resulting operator K. Toward this end it should be noted that the upper bound always holds if the operator J is bounded.

Lemma. *Let C be a self-adjoint operator with spectral resolution $C = \int \lambda\, dE_\lambda$. Let J be a bounded self-adjoint operator such that for some bounded subinterval Δ*

$$\langle JE(\Delta)\phi_1, CE(\Delta)\phi_1\rangle - \langle CE(\Delta)\phi_1, JE(\Delta)\phi_1\rangle = -i\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle.$$

Then $|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle| \leqslant \|J\|(|\Delta|)\|E(\Delta)\phi_1\|^2$.

Proof. Let λ be the midpoint of Δ. Then

$$\langle JE(\Delta)\phi_1, (C-\lambda I)E(\Delta)\phi_1\rangle - \langle (C-\lambda I)E(\Delta)\phi_1, JE(\Delta)\phi_1\rangle = -i\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle$$

and since $\|(C-\lambda I)E(\Delta)\phi_1\| = |\int_\Delta (x-\lambda)\,d\|E_x\phi_1\|^2| \leqslant \frac{1}{2}|\Delta|\|E(\Delta)\phi_1\|^2$ it follows that

$$\left|\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle\right| \leqslant 2\|J\|\left(\frac{1}{2}|\Delta|\right)\left\|E(\Delta)\phi_1\right\|^2.$$

Thus given a cyclic self-adjoint Jacobi operator C, with spectral decomposition $C = \int \lambda\, dE_\lambda$, and cyclic vector ϕ_1, the main issue for establishing the absolute continuity of the spectral measure on an interval I is to find a bounded operator J which leads to an operator K for which one can establish the needed lower bound for vectors of the form $E(\Delta)\phi_1$ where $\triangle$ is any bounded subinterval of I. □

3. Preliminary results

In the results that follow it will be assumed that the diagonal entries of the matrix operator C vanish, and that the subdiagonal entries are strictly positive. To establish the appropriate commutator equation a second (bounded) operator is needed. To this end, let J be defined by a positive bounded sequence $\{\alpha_n\}$ as follows:

$$J = \frac{1}{2i}\begin{bmatrix} 0 & -\alpha_1 & 0 & 0 & \dots \\ \alpha_1 & 0 & -\alpha_2 & 0 & \dots \\ 0 & \alpha_2 & 0 & -\alpha_3 & \dots \\ 0 & 0 & \alpha_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then $CJ - JC = -2iK$ where $K = [k_{ij}]$ with $k_{11} = \alpha_1 a_1$, and for $i > 1$, $k_{ii} = \alpha_i a_i - \alpha_{i-1}a_{i-1}$, $k_{i,i+2} = k_{i+2,i} = \frac{1}{2}(a_{i+1}\alpha_i - a_i\alpha_{i+1})$ and all other entries are zero. Thus with this choice for J the operator K has a distinctive five diagonal structure. The following lemmas present some interesting properties of such operators.

Lemma 3.1. *Assume that the real infinite matrix $K = [k_{ij}]$ has the following structure: $k_{ii} = s_i$, $k_{i,i+2} = k_{i+2,i} = t_i$, with all other entries equal to zero. If the sequence $\{t_i\}$ is bounded or if for all $i \geqslant N$, $|t_i| + |t_{i+1}| > 0$ and $\sum_{i=N}^\infty \frac{1}{|t_i|+|t_{i+1}|} = \infty$, then K, defined on the maximal domain $D_K = \{x \in \ell^2\colon\ Kx \in \ell^2\}$, is self-adjoint.*

Proof. For $x = \{x_n\}$ and $y = \{y_n\}$ in D_K,

$$\left|\langle Kx, y\rangle - \langle x, Ky\rangle\right| = \lim_{N\to\infty} |t_{N-1}x_{N+1}\overline{y}_{N-1} + t_N x_{N+2}\overline{y}_N - t_{N-1}\overline{y}_{N+1}x_{N-1} - t_N\overline{y}_{N+2}x_N|.$$

Since this limit exists it must equal zero. For if the limit is $p > 0$, then for large values of N,

$$\frac{p}{2} < \left(|t_{N-1}| + |t_N|\right)\left(|x_{N+1}\overline{y}_{N-1} - \overline{y}_{N+1}x_{N-1}| + |x_{N+2}\overline{y}_N - \overline{y}_{N+2}x_N|\right)$$

and either condition leads to a contradiction given that $\{x_n\}$ and $\{y_n\}$ are in ℓ^2, and hence $\sum_N |x_{N+1}\overline{y}_{N-1} - \overline{y}_{N+1}x_{N-1}| + |x_{N+2}\overline{y}_N - \overline{y}_{N+2}x_N| < \infty$. □

Lemma 3.2. *Assume that the real infinite matrix $K = [k_{ij}]$ satisfies the conditions of the previous lemma. If $t_i \geqslant 0$ for all i, and if $\exists M \geqslant 0$ such that $|t_{i-2} + s_i + t_i| \leqslant M$ for $i \geqslant 3$, then for $x = \{x_i\}$ in the domain of the operator K,*

$$\langle Kx, x\rangle = (s_1 + t_1)|x_1|^2 + (s_2 + t_2)|x_2|^2 + \sum_{i=3}^\infty (t_{i-2} + s_i + t_i)|x_i|^2 - \sum_{i=1}^\infty t_i|x_i - x_{i+2}|^2.$$

Proof. For $x = \{x_i\}$

$$\begin{aligned}\langle Kx, x\rangle &= (s_1 + t_1)|x_1|^2 + t_1(x_3 - x_1)\bar{x}_1 + (s_2 + t_2)|x_2|^2 + t_2(x_1 - x_2)\bar{x}_2 \\ &\quad + \lim_{n\to\infty}\sum_{i=3}^n (t_{i-2} + s_i + t_i)|x_i|^2 \\ &\quad + \lim_{n\to\infty}\left[\sum_{i=3}^n t_i(x_{i+2} - x_i)\bar{x}_i + \sum_{i=3}^n t_{i-2}(x_{i-2} - x_i)\bar{x}_i\right].\end{aligned}$$

It then follows that

$$\langle Kx, x\rangle = (s_1 + t_1)|x_1|^2 + (s_2 + t_2)|x_2|^2 + \lim_{n\to\infty}\left[\sum_{i=3}^{n}(t_{i-2} + s_i + t_i)|x_i|^2 - \sum_{i=1}^{n-2} t_i|x_{i+2} - x_i|^2 + t_{n-1}(x_{n+1} - x_{n-1})\bar{x}_{n-1} + t_n(x_{n+2} - x_n)\bar{x}_n\right].$$

This limit exists since $x \in D_K$ and $\sum_{i=3}^{\infty}(t_{i-2} + s_i + t_i)|x_i|^2$ is absolutely summable. Thus if $\sum_{i=1}^{\infty} t_i|x_{i+2} - x_i|^2 < \infty$, $\lim_{n\to\infty} |t_{n-1}(x_{n+1} - x_{n-1})\bar{x}_{n-1} + t_n(x_{n+2} - x_n)\bar{x}_n|$ exits, and the argument used in the proof of Lemma 3.1 shows that this limit must be zero. Note that if $\sum_{i=1}^{\infty} t_i|x_{i+2} - x_i|^2 = \infty$ then for some $q > 0$ and some integer N, $|t_{n-1}(x_{n+1} - x_{n-1})\bar{x}_{n-1} + t_n(x_{n+2} - x_n)\bar{x}_n| \geqslant q$ for $n \geqslant N$, which leads to a contradiction by a very similar argument. Thus it must be true that $\sum_{i=1}^{\infty} t_i|x_{i+2} - x_i|^2 < \infty$ and the result follows. □

The following observation was made in [12].

Lemma 3.3. *Assume that the real infinite matrix $K = [k_{ij}]$ has the following structure: $k_{ii} = s_i$, $k_{i,i+2} = k_{i+2,i} = t_i$. Then K is unitarily equivalent to $-K'$ where $K' = [k'_{ij}]$, $k'_{ii} = -s_i$, and $k'_{i,i+2} = k'_{i+2,i} = t_i$.*

Proof. Let U be the diagonal operator defined by $U\phi_{2n-1} = (-1)^{n+1}\phi_{2n-1}$, $U\phi_{2n} = (-1)^{n+1}\phi_{2n}$. It is easily verified that $UKU = -K'$. □

The next lemma, needed in the proof of the main results, refers to the self-adjoint Jacobi operator C. If the diagonal entries vanish then the spectral measure of C is symmetric about the origin. In the function space representation of the operator the corresponding basis vectors P_n are even or odd polynomials. This leads to the following useful observation from [6].

Lemma 3.4. *Assume that C is a self-adjoint Jacobi matrix with $a_n > 0$, $b_n = 0$, and spectral decomposition $C = \int \lambda\, dE_\lambda$. Let Δ be a bounded subinterval of $(0, \infty)$. If $x = E(\Delta)\phi_1 = \sum x_n\phi_n$ where $x_n = \langle E(\Delta)\phi_1, \phi_n\rangle$ then $\sum |x_{2n}|^2 = \sum |x_{2n-1}|^2$.*

Proof. For any bounded subinterval Δ of $(0, \infty)$, the corresponding spectral projections $E(\Delta)$ and $E(-\Delta)$ are orthogonal, and so $\langle E(\Delta)\phi_1, E(-\Delta)\phi_1\rangle = 0$. But $\langle E(-\Delta)\phi_1, \phi_n\rangle = \int_{-\Delta} P_n\, d\mu = (-1)^{n+1}\int_{\Delta} P_n\, d\mu = (-1)^{n+1}\langle E(\Delta)\phi_1, \phi_n\rangle$. Thus

$$0 = \bigl\langle E(\Delta)\phi_1, E(-\Delta)\phi_1\bigr\rangle = \sum \bigl|\bigl\langle E(\Delta)\phi_1, \phi_{2n-1}\bigr\rangle\bigr|^2 - \sum \bigl|\bigl\langle E(\Delta)\phi_1, \phi_{2n}\bigr\rangle\bigr|^2. \qquad \square$$

4. Main results

The following theorem offers a general strategy for obtaining results on absolute continuity for Jacobi operators defined by a positive monotone sequence $\{a_n\}$. It will be reformulated below in terms of conditions on the difference sequence $\{d_n\}$, where $d_n = a_n - a_{n-1}$, with $a_0 = 0$.

Theorem 4.1. *Let C be a self-adjoint Jacobi matrix with zero diagonal and subdiagonal entries $\{a_n\}$, $a_{n+1} \geqslant a_n > 0$. Suppose J is defined by a positive bounded sequence $\{\alpha_n\}$ so that if $CJ - JC = -2iK$ and $K = [k_{ij}]$ then $k_{i,i+2} \geqslant 0$. Let $D = [d_{ij}]$ be the diagonal operator with $d_{11} = -k_{11} + k_{13}$, $d_{22} = -k_{22} + k_{24}$, $d_{ii} = k_{i,i-2} - k_{ii} + k_{i,i+2}$ for $i > 2$. If $C = \int \lambda\, dE_\lambda$ and for some interval I, $I \subseteq (0, \infty)$, there exists $\alpha > 0$ such that for every bounded subinterval Δ of I, $\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\rangle \leqslant -\alpha\|E(\Delta)\phi_1\|^4$, then the spectral measure of C is absolutely continuous on I and $-I$.*

Proof. Obtain K_1 from K by negating the diagonal entries of K. Now choose a bounded subinterval Δ of I, and let $E(\Delta)\phi_1 = \sum x_n\phi_n$, where $x_n = \langle E(\Delta)\phi_1, \phi_n\rangle$. By Lemma 3.2,

$$\bigl\langle K_1E(\Delta)\phi_1, E(\Delta)\phi_1\bigr\rangle = \bigl\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\bigr\rangle - \sum_{n=1}^{\infty} k_{n,n+2}|x_n - x_{n+2}|^2.$$

By Lemma 3.3, $K = -UK_1U$, where U is the diagonal operator defined by $U\phi_{2n-1} = (-1)^{n+1}\phi_{2n-1}$ and $U\phi_{2n} = (-1)^{n+1}\phi_{2n}$. Then

$$\bigl\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\bigr\rangle = -\bigl\langle K_1UE(\Delta)\phi_1, UE(\Delta)\phi_1\bigr\rangle \geqslant -\bigl\langle DUE(\Delta)\phi_1, UE(\Delta)\phi_1\bigr\rangle = -\bigl\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\bigr\rangle.$$

Thus $\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle \geqslant \alpha\|E(\Delta)\phi_1\|^4$ which establishes the lower bound needed for the proof of the absolute continuity of the spectral measure on the interval I and, by symmetry, on $-I$. □

A sufficient (but not necessary) condition for the above theorem to hold is that the diagonal operator D satisfy the condition $D \leqslant 0$, $d_{11} < 0$. This gives a new proof of the following result which first appeared in [2].

Corollary 4.2. *Let C be a self-adjoint Jacobi matrix with zero diagonal and subdiagonal entries $\{a_n\}$, $a_{n+1} \geqslant a_n > 0$. Let $d_1 = a_1$, and $d_n = a_n - a_{n-1}$ for $n \geqslant 2$. If $\{d_n\}$ is bounded, and $d_n \geqslant \frac{1}{2}d_{n-1} + \frac{1}{2}d_{n+1}$ for $n \geqslant N \geqq 2$ then the spectral measure of C is absolutely continuous on $(-\infty, 0) \cup (0, \infty)$.*

Proof. Define J by letting $\alpha_i = a_i$ for $i = 1, \dots, N$ and $\alpha_n = a_n$ for $n \geqslant N$. If $CJ - JC = -2iK$ with $K = [k_{ij}]$ then the non-zero entries of K are $k_{11} = a_1^2$, $k_{ii} = a_i^2 - a_{i-1}^2$ for $i = 2, \dots, N$, $k_{ii} = a_N(a_i - a_{i-1})$ for $i > N$, and $k_{i,i+2} = k_{i+2,i} = \frac{1}{2}a_N(a_{i+1} - a_i)$ for $i \geqslant N$. If $D = [d_{ij}]$ is defined as in Theorem 4.1 then $d_{11} = -a_1^2$, $d_{nn} = -(a_n^2 - a_{n-1}^2)$ for $n = 2, \dots, N-1$, $d_{NN} = -(a_N^2 - a_{N-1}^2) + \frac{1}{2}a_N(a_{N+1} - a_N)$ and $d_{nn} = -a_N(a_n - a_{n-1}) + \frac{1}{2}a_N(a_{n-1} - a_{n-2}) + \frac{1}{2}a_N(a_{n+1} - a_n)$ for $n > N$. Hence for any bounded interval Δ contained in $(0, \infty)$, $\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\rangle \leqslant -a_1^2\|E(\Delta)\phi_1\|^4$ and the result follows. □

The previous corollary holds, for example, if for any non-negative constant r, $a_n = n + r$, $n \geqslant 1$.

For the special case that J is defined by $\alpha_i = 1$ for all i, Theorem 4.1 can be restated in terms of properties of the difference sequence $\{d_n\}$.

Theorem 4.3. *Let C be a self-adjoint Jacobi matrix with zero diagonal and subdiagonal entries $\{a_n\}$, $a_{n+1} \geqslant a_n > 0$. Define $d_1 = a_1$, and $d_n = a_n - a_{n-1}$ for $n \geqslant 2$, and assume that $\{d_n\}$ is bounded or that $\sum \frac{1}{|d_n|+|d_{n+1}|} = \infty$. Let $D = [d_{ij}]$ be the diagonal matrix with diagonal entries $d_{11} = -d_1 + \frac{1}{2}d_2$, $d_{22} = -d_2 + \frac{1}{2}d_3$, and $d_{nn} = -d_n + \frac{1}{2}d_{n-1} + \frac{1}{2}d_{n+1}$ for $n \geqslant 3$. Let $C = \int \lambda\, dE_\lambda$ and let $I \subseteq (0, \infty)$ be an interval. If for some $\alpha > 0$ and every bounded subinterval Δ of I, $\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\rangle \leqslant -\alpha\|E(\Delta)\phi_1\|^4$, then the spectral measure of C is absolutely continuous on I and $-I$.*

Proof. Since $a_n > 0$, ϕ_1 is a cyclic vector for C. Choose J as in Section 3 above with $\alpha_n = 1$ for $n \geqslant 1$. Compute K from the commutator equation $CJ - JC = -iK$. Then $K = [k_{ij}]$, where $k_{ii} = d_i$, $k_{i,i+2} = k_{i+2,i} = \frac{1}{2}d_{i+1}$ and all other entries equal zero. By Lemma 3.1 K is self-adjoint. Obtain K_1 from K by negating the diagonal entries of K. Then the nth diagonal entry of D is the sum of the entries in the nth row of K_1. Now choose a bounded subinterval Δ of I, and let $E(\Delta)\phi_1 = \sum x_n\phi_n$, where $x_n = \langle E(\Delta)\phi_1, \phi_n\rangle$. By Lemma 3.2,

$$\big\langle K_1E(\Delta)\phi_1, E(\Delta)\phi_1\big\rangle = \big\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\big\rangle - \sum_{n=1}^{\infty}\frac{1}{2}d_n|x_n - x_{n+2}|^2.$$

By Lemma 3.3, $K = -UK_1U$, where U is the diagonal operator defined by $U\phi_{2n-1} = (-1)^{n+1}\phi_{2n-1}$ and $U\phi_{2n} = (-1)^{n+1}\phi_{2n}$. Then

$$\big\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\big\rangle = -\big\langle K_1UE(\Delta)\phi_1, UE(\Delta)\phi_1\big\rangle \geqslant -\big\langle DUE(\Delta)\phi_1, UE(\Delta)\phi_1\big\rangle = -\big\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\big\rangle.$$

Thus $\langle KE(\Delta)\phi_1, E(\Delta)\phi_1\rangle \geqslant \alpha\|E(\Delta)\phi_1\|^4$ which establishes the lower bound needed for the proof of the absolute continuity of the spectral measure on the interval I and, by symmetry, on $-I$. □

The following corollary shows that D need not be a negative operator. An alternate proof of the following result can be found in [7].

Corollary 4.4. *Let C be a self-adjoint Jacobi matrix with zero diagonal and subdiagonal entries $\{a_n\}$, $a_{n+1} \geqslant a_n > 0$. Define $d_1 = a_1$, and $d_n = a_n - a_{n-1}$ for $n \geqslant 2$, and assume that for $\delta > 0$, $d > 0$, $d_{2n} = \delta$ and $d_{2n+1} = d$ for $n \geqslant 1$. If $a_1 + \frac{\delta}{2} - d > 0$ the spectral measure of C is absolutely continuous on $(-\infty, 0) \cup (0, \infty)$. If $a_1 + \frac{\delta}{2} - d \leqslant 0$ then the spectral measure of C is absolutely continuous on $(-\infty, -z) \cup (z, \infty)$ where z satisfies the condition $(a_1 + \frac{\delta}{2} - d) + \frac{1}{2}d\frac{z^2}{a_1^2} > 0$.*

Proof. Let J, K, K_1, and D be defined as in the proof of Theorem 4.3. Then $D = [d_{ij}]$ where $d_{11} = -d_1 + \frac{1}{2}\delta = (\delta - d) - (a_1 + \frac{\delta}{2} - d)$, $d_{22} = (d - \delta) - \frac{d}{2}$, $d_{2i-1,2i-1} = \delta - d$, $d_{2i,2i} = d - \delta$, $i > 1$. For any bounded subinterval Δ of $(0, \infty)$, let $E(\Delta)\phi_1 = \sum x_n\phi_n$, where $x_n = \langle E(\Delta)\phi_1, \phi_n\rangle$. Since $\sum |x_{2n}|^2 = \sum |x_{2n-1}|^2$ it follows that

$$\big\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\big\rangle = -\left(a_1 + \frac{\delta}{2} - d\right)|x_1|^2 - \frac{d}{2}|x_2|^2.$$

Thus if $a_1 + \frac{\delta}{2} - d > 0$ the spectral measure of C is absolutely continuous on $(-\infty, 0) \cup (0, \infty)$. Otherwise, recall that $x_2 = \int_\Delta P_2\, d\mu$ and $P_2(x) = \frac{x}{a_1}$. Thus

$$\left\langle DE(\Delta)x, E(\Delta)x\right\rangle \leqslant -\left(a_1 + \frac{\delta}{2} - d\right)\left\|E(\Delta)\phi_1\right\|^4 - \frac{d}{2}\left(\frac{z^2}{a_1^2}\right)\left\|E(\Delta)\phi_1\right\|^4,$$

and the result follows from Theorem 4.3. □

Example. Let $a_{2n+1} = a_{2n} = n$. Then $d_{2n-1} = 1$, $d_{2n} = 0$. In this case it is shown in [11] that $(-\frac{1}{2}, \frac{1}{2})$ is a gap in the spectrum. Corollary 4.4 provides another way of establishing the absolute continuity of the spectral measure spectral outside the spectral gap.

Example. Choose $x > -1$, $y > -2$ so that $|x - y| < 1$. Let $a_{2n-1} = (2n-1) + x$, $a_{2n} = (2n) + y$ for $n \geqslant 1$. Then $\delta = 1 + y - x$ and $d = 1 + x - y$. In this case it is shown in [5] that $(-|x-y|, |x-y|)$ is a gap in the essential spectrum. Corollary 4.4 provides criteria for absolute continuity. If $y > 0$, for example, then the spectral measure is absolutely continuous on $(-\infty, 0) \cup (0, \infty)$.

Example. Choose $0 < \omega \leqslant 1$, $\rho > 0$. Let $d_{2n} = \delta = 1 - \omega$ and $d_{2n+1} = d = 1 + \omega + \rho$. In this case it was shown in [4] that $(-\omega - \frac{\rho}{2}, \omega + \frac{\rho}{2})$ is a gap in the essential spectrum. Corollary 4.4 again provides criteria for establishing absolute continuity. Note that in this case $a_n = n + c_n$ where the perturbation sequence $\{c_n\}$ is unbounded. However, the difference sequence $\{d_n\}$ is bounded.

Whereas the previous examples deal with classes of Jacobi operators from the literature and, in some cases, apply the techniques of this paper to obtain new proofs of previously observed results, the following theorem applies the techniques of this paper to a new class of Jacobi operators for which the difference sequence is unbounded.

Theorem 4.5. *Let C be a Jacobi matrix with zero diagonal and subdiagonal entries $\{a_n\}$, $a_{n+1} \geqslant a_n > 0$. Define $d_1 = a_1$, and $d_n = a_n - a_{n-1}$ for $n \geqslant 2$, and assume that for $r \geqslant 0$, $d_{2n} = \delta + nr$, $d_{2n+1} = d + nr$, $n \geqslant 1$. Then C is self-adjoint. If $a_1 + \frac{\delta}{2} - d > 0$ the spectral measure of C is absolutely continuous on $(-\infty, 0) \cup (0, \infty)$. If $a_1 + \frac{\delta}{2} - d \leqslant 0$ then the spectral measure of C is absolutely continuous on $(-\infty, -z) \cup (z, \infty)$ where z satisfies the condition $(a_1 + \frac{\delta}{2} - d) + \frac{1}{2}d\frac{z^2}{a_1^2} > 0$.*

Proof. It was shown in [6] that if $\sum_{n=1}^{\infty}(\frac{a_1a_3\cdots a_{2n-1}}{a_2a_4\cdots a_{2n}})^2 < \infty$ and $\sum_{n=1}^{\infty}(\frac{a_2a_4\cdots a_{2n}}{a_3a_5\cdots a_{2n+1}})^2 = \infty$ then C is self-adjoint and 0 is an eigenvalue for C. To verify these conditions note that for $n \geqslant 1$, $a_{2n} = a_1 + n\delta + (n-1)d + n(n-1)r$ and $a_{2n+1} = a_1 + n\delta + nd + n^2r$. If $q_n = (\frac{a_2\cdots a_{2n+2}}{a_3\cdots a_{2n+1}})^2$ then $\frac{q_{n+1}}{q_n} = \frac{a_{2n+2}^2}{a_{2n+1}^2} = (\frac{a_1 + (n+1)\delta + nd + (n+1)nr}{a_1 + n\delta + nd + n^2r})^2$ and an application of Gauss's Test (see [10]) shows that $\sum_{n=1}^{\infty}(\frac{a_1\cdots a_{2n}}{a_3\cdots a_{2n+1}})^2 = \infty$. Similarly, let $v_n = (\frac{a_1\cdots a_{2n-1}}{a_2\cdots a_{2n}})^2$. Then $\frac{v_{n+1}}{v_n} = (\frac{a_{2n+1}}{a_{2n+2}})^2 = (\frac{a_1 + n\delta + nd + n^2r}{a_1 + (n+1)\delta + nd + (n+1)nr})^2$ and in this case it follows from Gauss's Test that $\sum_{n=1}^{\infty}(\frac{a_1\cdots a_{2n-1}}{a_2\cdots a_{2n}})^2 < \infty$. Thus C is self-adjoint and 0 is an eigenvalue.

To establish absolute continuity on subintervals of $(-\infty, 0) \cup (0, \infty)$, note that for $n \geqslant 1$, $d_{2n+1} - \frac{1}{2}d_{2n} - \frac{1}{2}d_{2n+2} = (d - \delta) - \frac{1}{2}r$, and $d_{2n+2} - \frac{1}{2}d_{2n+1} - \frac{1}{2}d_{2n+3} = (\delta - d) + \frac{1}{2}r$. Let J, K, K_1, and D be defined as in the proof of Theorem 4.3. Then $D = [d_{ij}]$ where $d_{11} = -d_1 + \frac{1}{2}(\delta + r) = (\delta - d + \frac{1}{2}r) - (a_1 + \frac{\delta}{2} - d)$, $d_{22} = (d - \delta - \frac{1}{2}r) - \frac{d}{2}$, $d_{2i-1,2i-1} = \delta - d + \frac{1}{2}r$, $d_{2i,2i} = d - \delta - \frac{1}{2}r$, $i > 1$. For any bounded subinterval Δ of $(0, \infty)$, let $E(\Delta)\phi_1 = \sum x_n\phi_n$, where $x_n = \langle E(\Delta)\phi_1, \phi_n\rangle$. Since $\sum |x_{2n}|^2 = \sum |x_{2n-1}|^2$ it follows that

$$\left\langle DE(\Delta)\phi_1, E(\Delta)\phi_1\right\rangle = -\left(a_1 + \frac{\delta}{2} - d\right)|x_1|^2 - \frac{d}{2}|x_2|^2.$$

Thus if $(a_1 + \frac{\delta}{2} - d) > 0$, the spectral measure is absolutely continuous on $(-\infty, 0)\bigcup(0, \infty)$. Otherwise, as in the proof of Corollary 4.4, the spectral measure is absolutely continuous on $(-\infty, z) \cup (z, \infty)$ where z satisfies the condition $(a_1 + \frac{\delta}{2} - d) + \frac{d}{2}\frac{z^2}{a_1^2} > 0$. □

Remarks. It is interesting to compare the results of Theorem 4.5 with some of the results in the literature on pure discrete spectrum. Sufficient conditions for a discrete spectrum are known for Jacobi operators for which the diagonal sequence $\{b_n\}$ satisfies the condition $\lim_{n\to\infty} |b_n| = \infty$. It is known, for example, that if C has diagonal sequence $\{b_n\}$ and subdiagonal sequence $\{a_n\}$ and if $\lim_{n\to\infty} |b_n| = \infty$ and $\lim_{n\to\infty} \frac{a_n^2}{b_nb_{n+1}} < \frac{1}{4}$, then C has a pure point spectrum. Also if $\lim_{n\to\infty} |b_n| = \infty$ and $\underline{\lim}_{n\to\infty} \frac{b_n^2}{a_n^2 + a_{n-1}^2} > 2$ then C has a discrete spectrum. Let C be a Jacobi matrix which satisfies the conditions of Theorem 4.5 (with diagonal entries equal to zero). Then C^2 restricted to the invariant subspace spanned by $\{\phi_{2n-1}\}$ is a Jacobi matrix for which the diagonal entries are positive and tend toward infinity. Denote the diagonal entries of this matrix by $\{\beta_n\}$ and the subdiagonal entries by $\{\alpha_n\}$. Then $\lim_{n\to\infty} \frac{\alpha_n^2}{\beta_n\beta_{n+1}} = \frac{1}{4}$ and $\lim_{n\to\infty} \frac{\beta_n^2}{\alpha_n^2 + \alpha_{n-1}^2} = 2$.

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